Session 5: Mechanics of Plasticity

Jennifer Hay
Factory Application Engineer
Nano-Scale Sciences Division
Agilent Technologies

jenny.hay@agilent.com

Motivation

Understanding the mechanics of plasticity helps us to know:

- The parameters which govern the onset of plasticity.
- The location in the material at which plasticity begins.

Exemplary test applications:

- For given materials and geometry, what is the maximum penetration depth for an elastic contact?
- For given materials, how sharp must the indenter tip be in order to measure hardness at a depth of 10nm?

Approach

- Use elastic contact mechanics to express the stresses which arise from contact.
- Plasticity ensues where and when the priniciple shear stress (calculated by elastic contact mechanics) exceeds half yield stress of the material in simple compression. (This is the Tresca criterion).

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"Refined experiments on metals specimens, carefully controlled to be isotropic, support the von Mises criterion of yielding. However, the difference in the predictions of the two criteria is not large and is hardly significant … It is justifiable, therefore, to employ Tresca's criterion where its algebraic simplicity makes it easier to use."

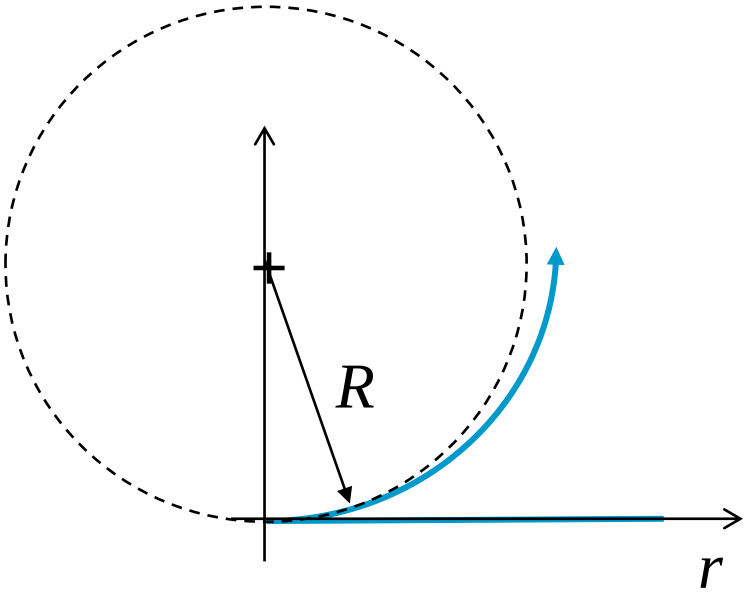
- Ken Johnson, *Contact Mechanics* (1985), p. 153.

Two ideal geometries

- **Hertzian** (paraboloid contacting a flat surface). Analysis is applied to spherical indenters AND the apex of sharp indenters (which are inevitably rounded to some extent).
- **Sharp** (self-similar cone contacting a flat surface). Analysis is applied to sharp indenters when apical radius is small relative to displacement.

- Note: These are IDEALS. The ideal we choose as representative finally depends on the application.

Hertzian contacts



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Hertzian contacts

$$P = \frac{4}{3} E_r R^{1/2} h^{3/2}$$

(Session 2)

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$$p_m = \frac{4}{3\pi} \frac{E_r}{R^{1/2}} h^{1/2}$$

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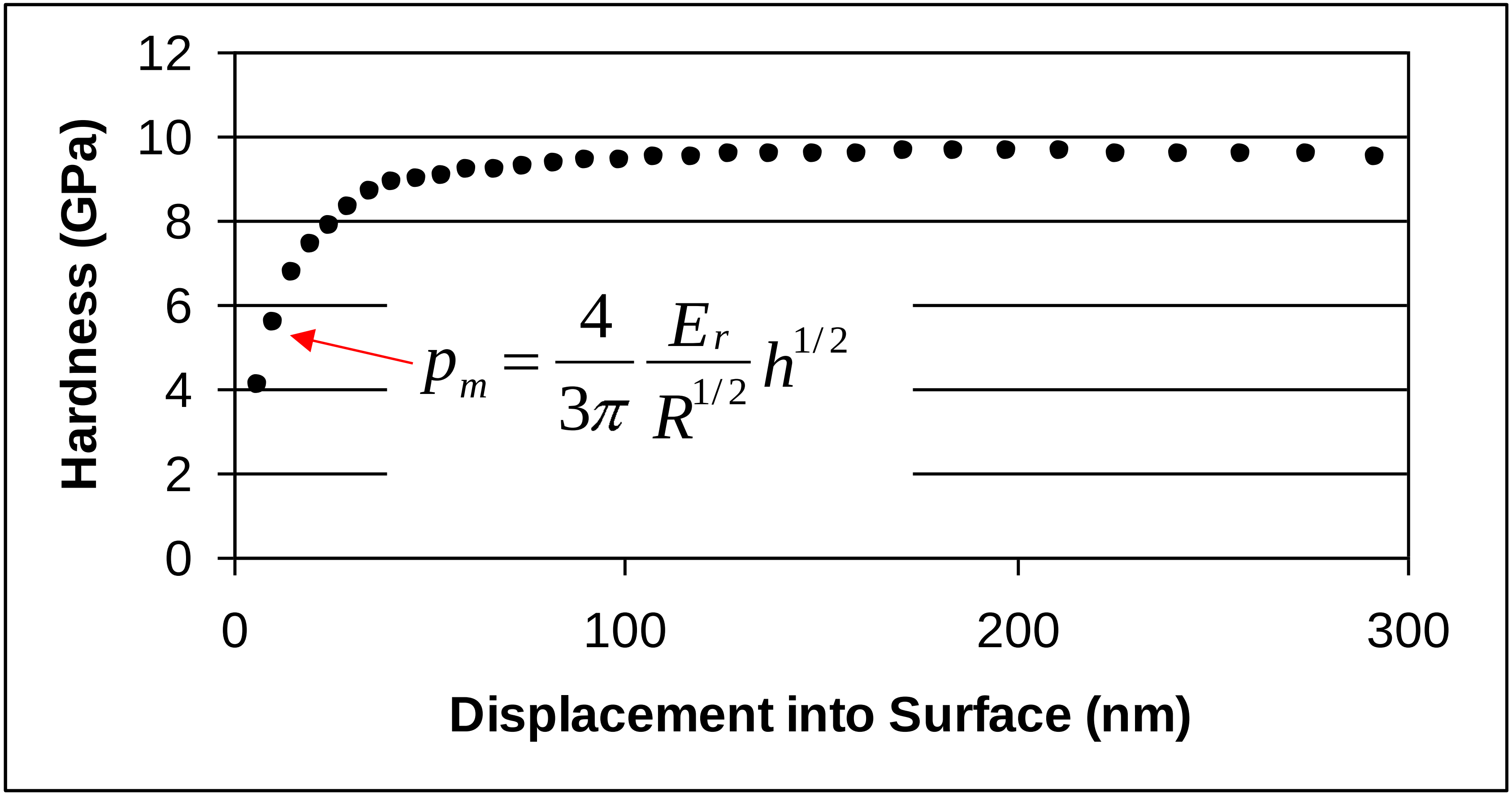
If "hardness" is defined as the mean contact pressure AND the contact is Hertzian, then:

- "Hardness" depends only on reduced modulus, tip radius (R), and displacement (h).
- "Hardness" goes to zero as displacement goes to zero.

Common plot of hardness of fused silica



Instrumented indentation results for fused silica
(NanoIndenter XP, Berkovich indenter, CSM)

Hertzian relation between max pressure (p_o) and mean pressure (p_m)

Hertz showed that the normal pressure at the surface is given by:

$$\sigma_{zz}(r, 0) = p_0\,[1-(r/a)^2]^{1/2}$$

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where p_o is the maximum normal pressure (at the center). To find the relation between the maximum normal pressure and the mean stress, we integrate the pressure distribution to get the load, and then divide by the area:

$$P = \int_0^a \sigma_{zz}(r,\ 0)\ \cdot 2\pi r\ dr = \frac{2}{3} p_0 \pi a^2$$

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$$p_m = \frac{2}{3} p_0$$

Maximum shear stress along the axis of symmetry

Along the axis of symmetry, Johnson calculates the principle shear stress to be (p. 60-62):

$$\tau_1 = \frac{1}{2}\left(\sigma_r - \sigma_z\right) = \frac{p_o}{2}\left[-\left(1+\nu\right)\left\{1 - \frac{z}{a}\tan^{-1}\frac{a}{z}\right\} + \frac{3}{2}\frac{1}{\left(1+z^2/a^2\right)}\right]$$

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$$\frac{\mathrm{d}\tau_1}{\mathrm{d}z} = 0$$

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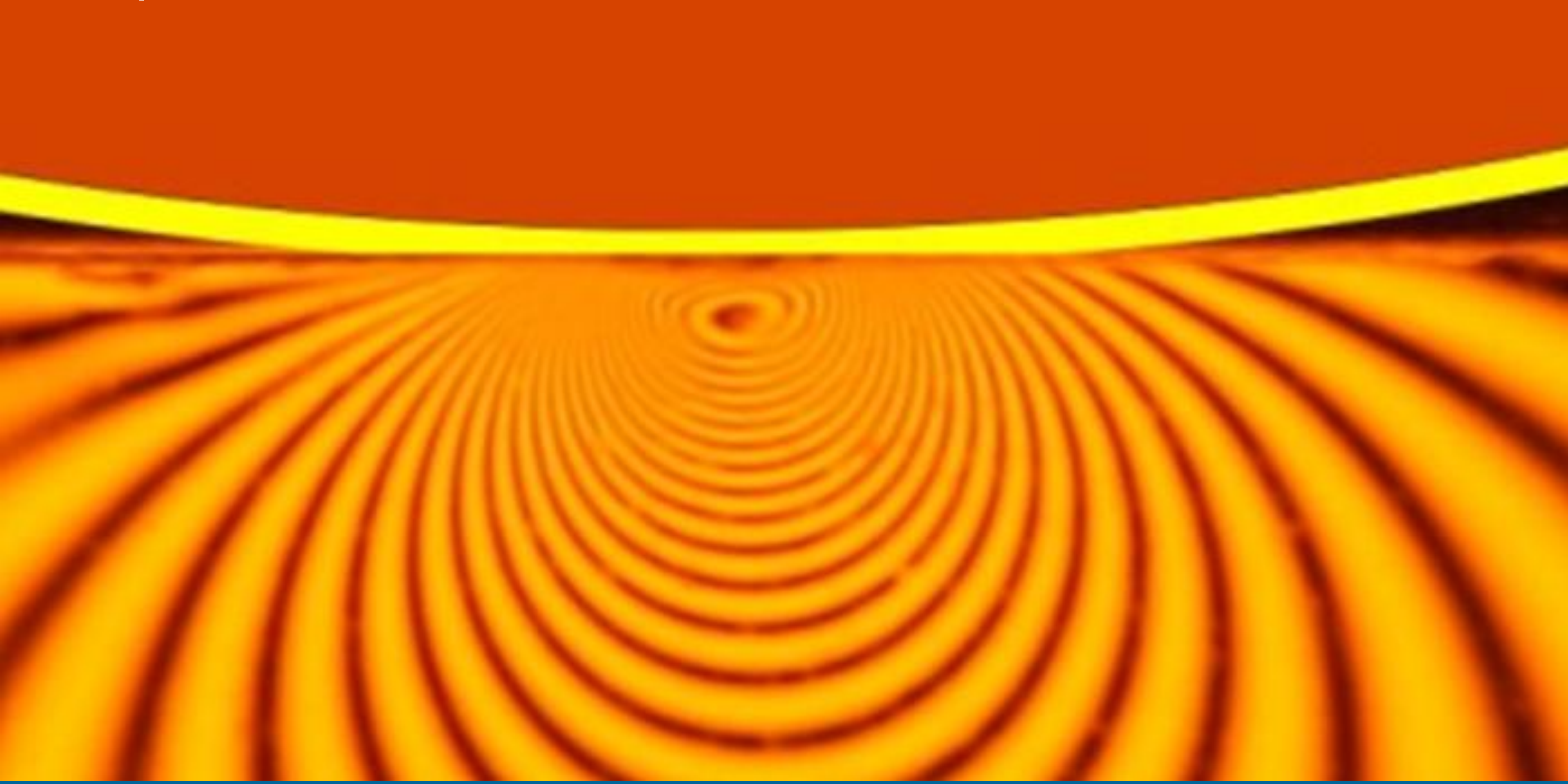
$$\frac{\mathrm{d}\tau_1}{\mathrm{d}z} = 0 \quad \Rightarrow \quad z = 0.57a$$

Location of maximum stress

Photoelastic image from http://upload.wikimedia.org/wikipedia/commons/1/18/Kontakt_Spannungsoptik.JPG (public domain)



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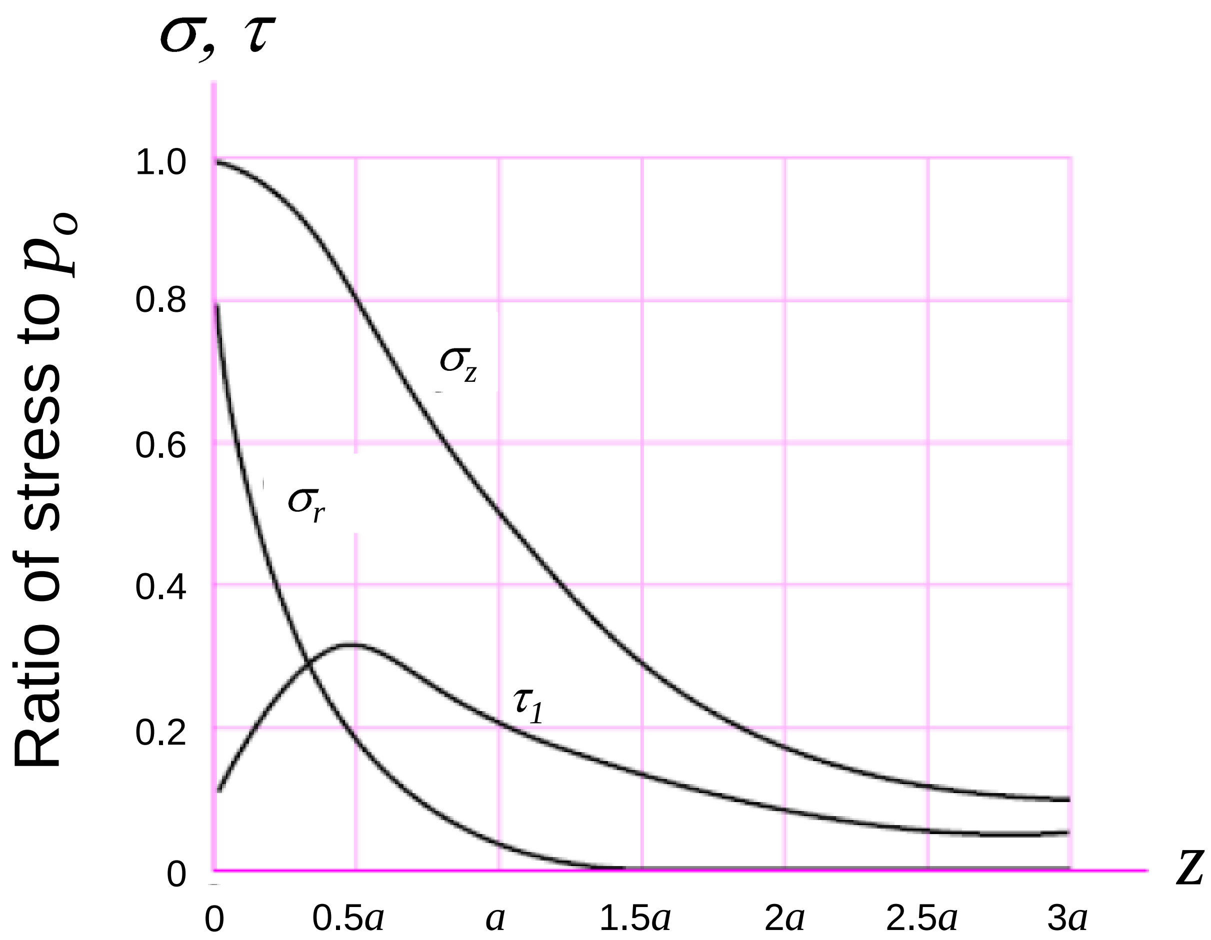
To find the position, z, of maximum shear stress, assume a typical Poisson's ratio of $\nu = 0.3$ and solve:

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}z} = 0 \Rightarrow z = 0.57a$$

$$\tau_{\max} = 0.31 p_o = 0.47 p_m$$

Hertzian stresses along the axis of symmetry



From an on-line tutorial: http://highered.mcgraw-hill.com/sites/dl/free/0072520361/82833/Ch04_Section20_Hertz_Contact_Stresses.pdf

For Hertzian contacts, remember the "halves":

- The maximum shear stress occurs below the surface by **half** the contact radius:

$$z = 0.57a$$

- The maximum shear stress has a value of about **half** the mean pressure:

$$\tau_{\max} = 0.47\, p_m$$

Apply the Tresca criterion:

Plasticity ensues when the shear stress exceeds half yield stress of the material in simple compression.

$$\tau_{\max} \geq \frac{\sigma_y}{2}$$

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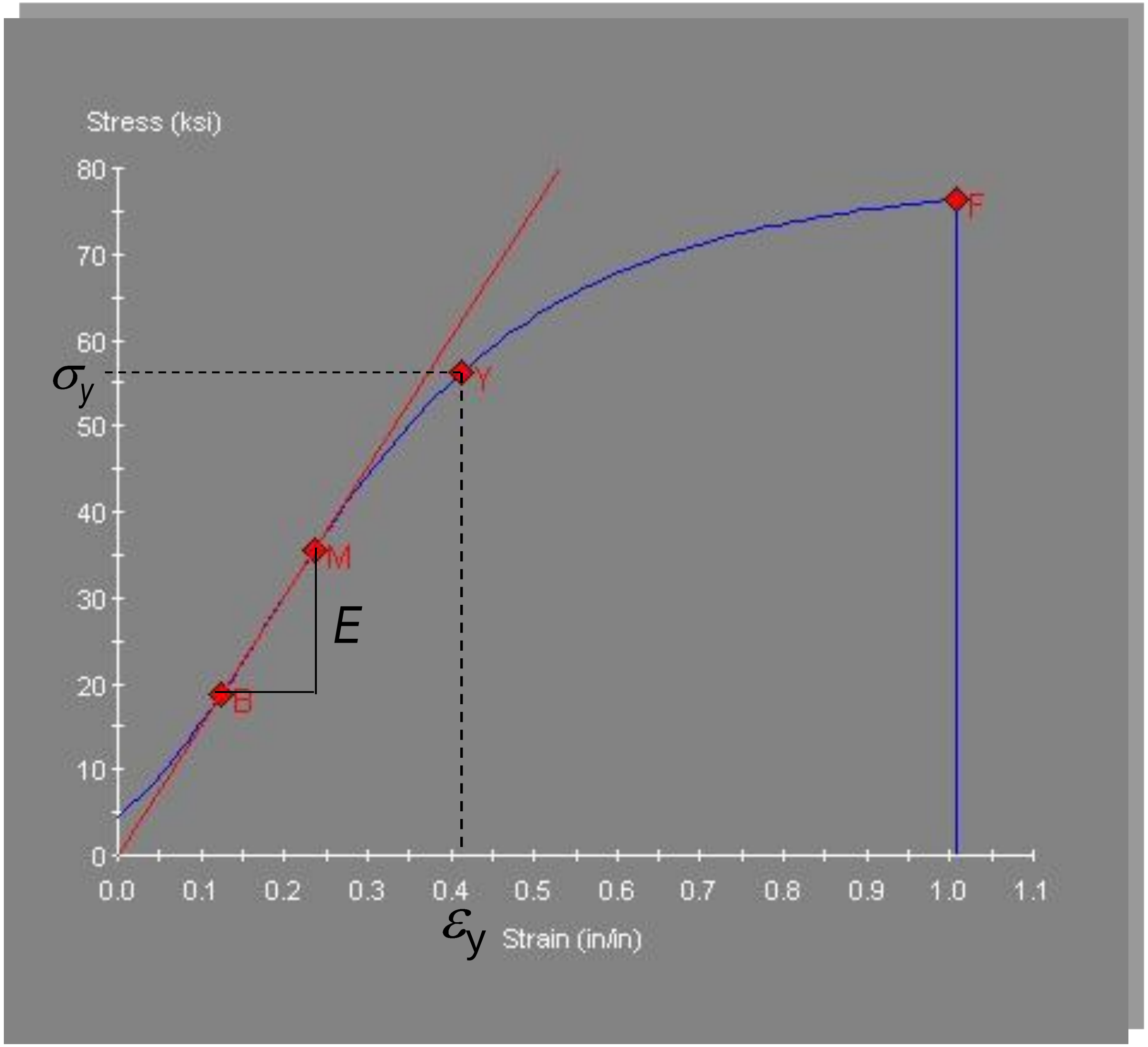
$$p_m = \frac{4}{3\pi} \frac{E_r}{R^{1/2}} h^{1/2} \quad \Rightarrow \quad \frac{4}{3\pi} \frac{E_r}{R^{1/2}} h^{1/2} \geq \sigma_y$$

For Hertzian contacts, yield occurs when

$$\frac{4}{3\pi}\frac{h^{1/2}}{R^{1/2}} \geq \frac{\sigma_y}{E_r}$$

“Yield Strain”



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Like the yield strain for the material in simple tension!

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So the left-hand side must also be a strain.

Like the yield strain for the material in simple tension!

Example: Maintaining elasticity

For given materials and contact geometry, what is the maximum penetration depth for an elastic contact?

Application: "Modulus mapping" by scanning. Contact area can only be calcuated if contact is elastic.

- In this application, we want to AVOID plasticity.
- Use the Hertzian criterion (not the sharp criterion) because apical rounding is not only inevitable, but desirable in this context. We must plan to work in the regime where apical rounding dominates the geometry.

Example: Maintaining elasticity

$$\frac{4}{3\pi}\frac{h^{1/2}}{R^{1/2}} \leq \frac{\sigma_y}{E_r}$$

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$$\frac{4}{3\pi}\frac{h^{1/2}}{R^{1/2}} \leq \frac{\sigma_y}{E_r}$$

Solve the inequality for h:

$$h \leq \frac{9\pi^2 R}{16}\left(\frac{\sigma_y}{E_r}\right)^2$$

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Example: Maximum displacement to maintain elasticity on nickel

$$h \le \frac{9\pi^2 R}{16}\left(\frac{\sigma_y}{E_r}\right)^2$$

Assume:

- Diamond Berkovich indenter
- Apical radius of R = 100nm (relatively blunt for a Berkovich).
- Properties of diamond:
  - E = 1140 GPa; ν = 0.07
- Properties of nickel:
  - E = 200 GPa; ν = 0.31; σ_y = 2GPa

Step 1: Calculate reduced modulus

$$\frac{1}{E_r} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

$$E_r = \left[\frac{1-0.07^2}{1140} + \frac{1-0.31^2}{200}\right]^{-1} = 185 GPa$$

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Step 2: Calculate maximum displacement

$$h \le \frac{9\pi^2(100nm)}{16}\left(\frac{2}{185}\right)^2$$

$$h \le 0.065nm$$

In order to maintain elastic contact between a diamond Berkovich indenter (R = 100nm) and nickel, the indentation depth must be less than an Angstrom!

For more analysis of this kind, read: http://cp.literature.agilent.com/litweb/pdf/5990-6329EN.pdf

Example: Achieving plasticity

For given materials, how sharp must the indenter tip be in order to measure hardness at a depth of 10nm?

- In this application, we want to ACHIEVE plasticity, because one cannot measure a hardness (mean pressure) that relates to yield stress without actually causing yield.

- Use the Hertzian criterion (not the sharp criterion), because we want to know how small the tip radius must be.

Example: Achieving plasticity

$$\frac{4}{3\pi}\frac{h^{1/2}}{R^{1/2}} \geq \frac{\sigma_y}{E_r}$$

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$$\frac{4}{3\pi}\frac{h^{1/2}}{R^{1/2}} \geq \frac{\sigma_y}{E_r}$$

Solve the inequality for R:

$$R \leq \frac{16h}{9\pi^2}\left(\frac{E_r}{\sigma_y}\right)^2$$

Example: Maximum tip radius to cause plasticity in fused silica at 10nm

$$R \leq \frac{16h}{9\pi^2}\left(\frac{E_r}{\sigma_y}\right)^2$$

Assume:

- h = 10nm
- Diamond Berkovich indenter
- Properties of diamond:
  - E = 1140 GPa; ν = 0.07
- Properties of fused silica:
  - E = 72 GPa; ν = 0.18; σ_y = 4GPa

Step 1: Calculate reduced modulus

$$\frac{1}{E_r} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

$$E_r = \left[\frac{1-0.07^2}{1140} + \frac{1-0.18^2}{72}\right]^{-1} = 69.9 GPa$$

Step 2: Calculate maximum tip radius

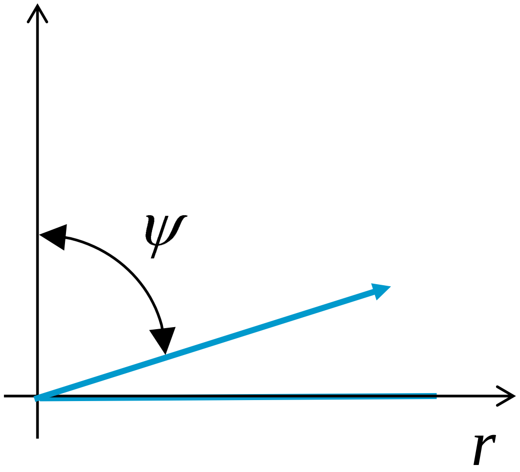
$$R \leq \frac{16(10nm)}{9\pi^2}\left(\frac{69.9}{4}\right)^2$$

$$R \leq 550nm$$

In order to cause plasticity by pressing a Berkovich indenter into fused silica to 10nm, the tip radius must be less than 550nm.

Note: Necessary tip radius goes with displacement. To initiate plasticity at 5nm, the tip radius must be less than 225nm.

Sharp contacts



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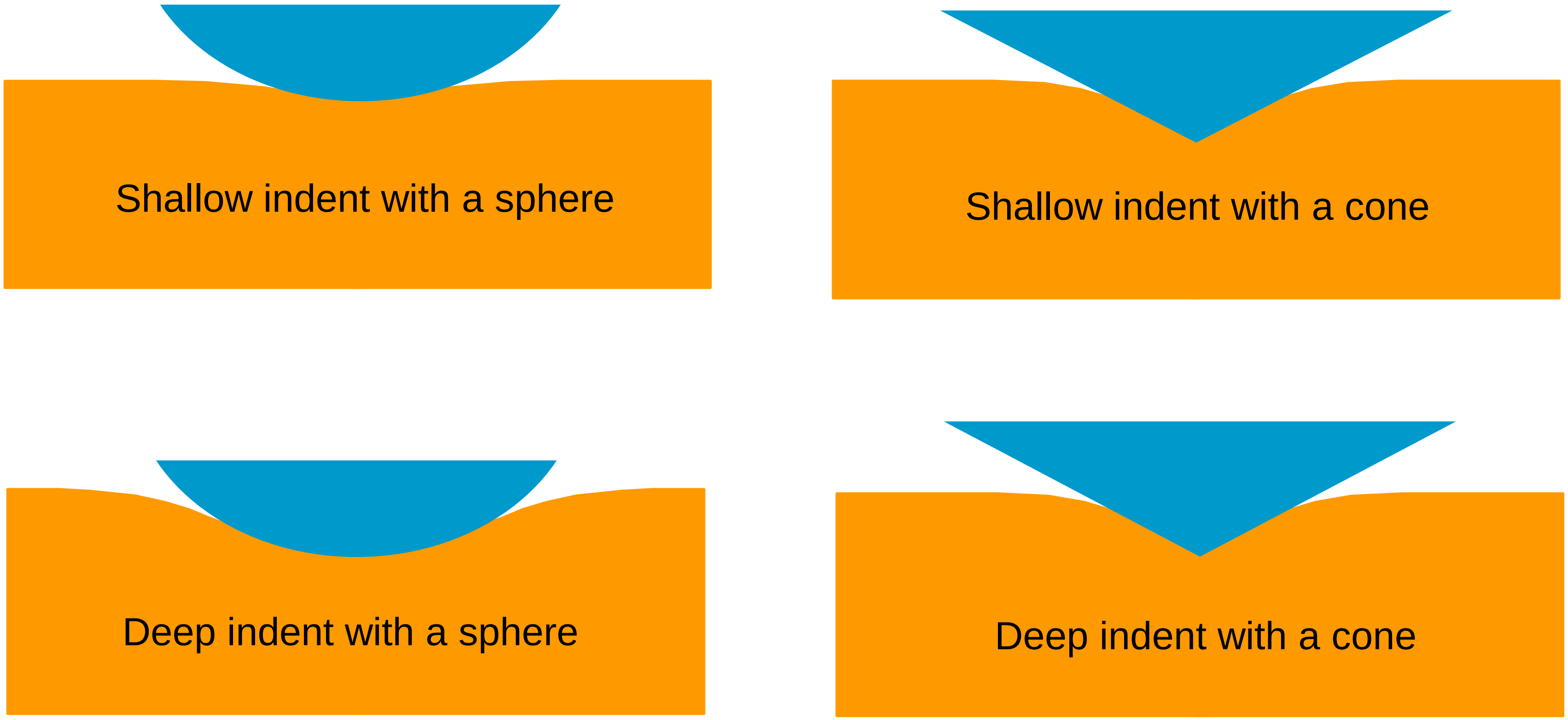
Indentation testing strains the material



Maximum shear stress along the axis of symmetry

Along the axis of symmetry, Johnson calculates the principle shear stress **for incompressible materials** ($\nu = 0.5$) to be (p. 111-114):

$$\tau_1 = \frac{1}{2}\left(\sigma_r - \sigma_z\right) = \frac{E_r a^2 \cot\psi}{2\left(a^2 + z^2\right)}$$

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As expected, the principle shear stress is largest at the apex (z = 0). At this position, it has a value of:

$$\tau_{max} = \frac{1}{2} E_r \cot\psi$$

Apply the Tresca criterion:

Plasticity ensues when and where the principle shear stress exceeds half yield stress of the material in simple tension (p. 156).

$$\tau_{\max} \geq \frac{\sigma_y}{2}$$

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Apply the Tresca criterion:

Plasticity ensues when and where the principle shear stress exceeds half yield stress of the material in simple tension (p. 156).

$$\tau_{\max} \geq \frac{\sigma_y}{2}$$

$$\tau_{\max} = \frac{1}{2} E_r \cot\psi \quad \Rightarrow \quad E_r \cot\psi \geq \sigma_y$$

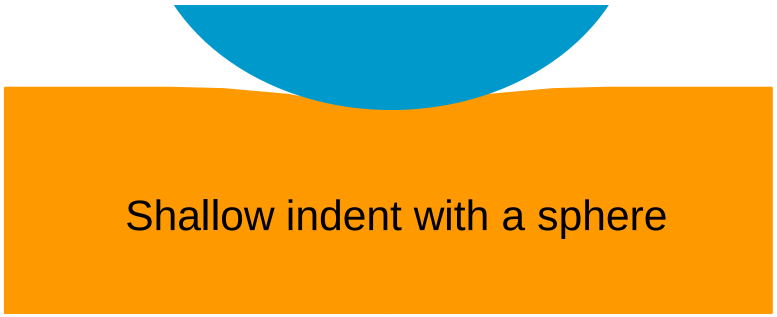
For sharp contacts, yield initiates when

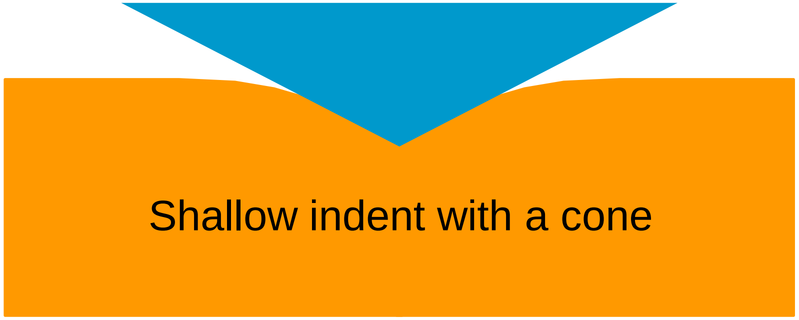
$$\cot\psi \geq \frac{\sigma_y}{E_r}$$

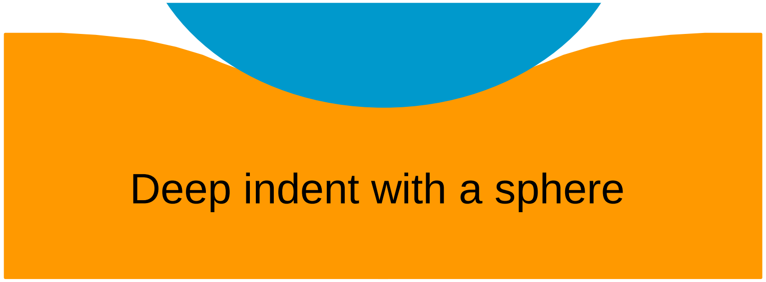
- Plasticity is governed only by the geometry - not by applied load or mean pressure. Yield occurs as soon as the indenter touches the material, or NOT AT ALL.

- You can picture $\cot\psi$ as the contact depth divided by the contact radius, which increases as the cone angle becomes smaller (i.e. as the cone becomes sharper).

Indentation testing strains the material







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For sharp contacts, yield initiates when

$$\cot\psi \geq \frac{\sigma_y}{E_r}$$

"For compressible materials, the results obtained above are no longer true…the infinite elastic pressure at the apex will give rise to theoretically infinite difference in principle stresses which will cause plastic flow however small the wedge or cone angle. Nevertheless, the plastic deformation arising in this way will, in fact, be very small and confined to a small region close to the apex…It would seem to be reasonable, therefore, to neglect the small plastic deformation which arises in this way and to retain [the above] equation to express the effective initiation of yield... (Johnson, p. 156)"

For sharp contacts, yield initiates when

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Like the yield strain for the material in simple tension!

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So the left-hand side must also be a strain.

Like the yield strain for the material in simple tension!

Summary for the onset of plastic yield

For Hertzian contacts, yield initiates below the contact surface (z = 0.5a) when the following criteria is met:

$$\frac{4\gamma}{3\pi}\frac{h^{1/2}}{R^{1/2}} \geq \frac{\sigma_y}{E_r}$$

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Summary for the onset of plastic yield

For Hertzian contacts, yield initiates below the contact surface (z = 0.5a) when the following criteria is met:

$$\frac{4\gamma}{3\pi}\frac{h^{1/2}}{R^{1/2}} \geq \frac{\sigma_y}{E_r}$$

For sharp contacts, yield initiates at z = 0 when the following criteria is met:

$$\cot\psi \geq \frac{\sigma_y}{E_r}$$

Full Plasticity

Full plasticity

- Up to this point, we have only considered the onset of plasticity.
- But for Hertzian contacts, plasticity initiates below the surface ($z \sim 0.5a$). For sharp contacts, plasticity initiates at the apex of the indenter, but plastically deformed material is constrained by indenter and elastic material of the sample.
- Initially, plastically deforming material is entirely surrounded by elastic material.
- Phenomenologically, “full plasticity” is achieved when plastically deforming material is no longer constrained by elastic material.

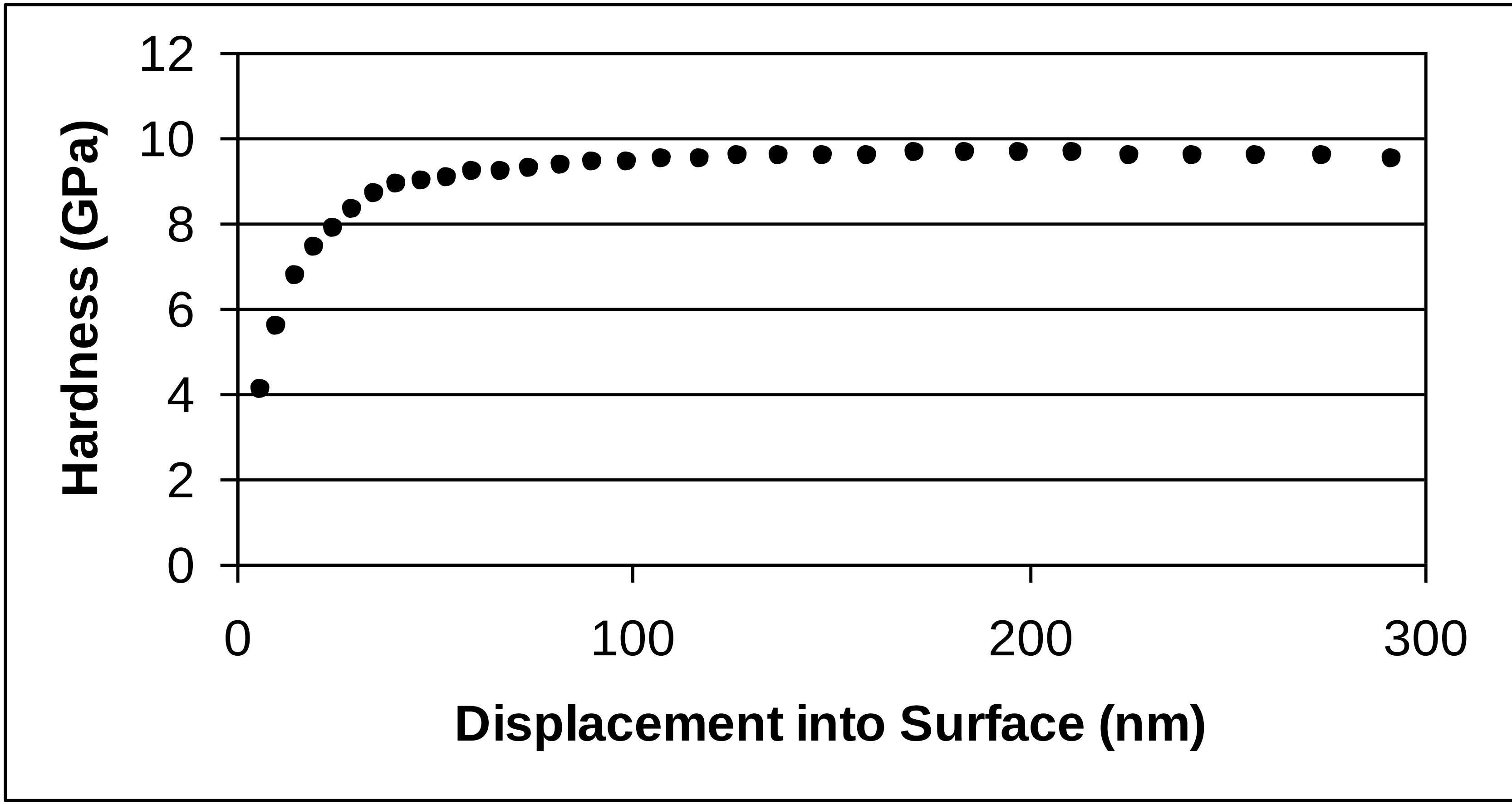
Full plasticity - continued

- When full plasticity is achieved, mean pressure (defined as “hardness”) depends strongly on yield stress.
- The relationship $H=C\sigma_y$ has been demonstrated by experiment and finite-element simulation for a variety of materials. The constant of proportionality, C, also known as the *constraint factor*, is about 2.8 ~ 3.2 for most metals, and is similar for both sharp and spherical indenters
- So an empirical definition for “full plasticity” is that point at which the hardness has a constant relationship with yield stress.

Full Plasticity - continued

- It is possible to achieve $H=C\sigma_y$ even if the plastic zone does not reach the free surface.
- Hardness will be a constant multiple of yield stress if the contact is *fully developed*; that is, if the volume of the plastic zone continues to grow proportionally with the volume of the indent.
- If the contact is fully developed, but not fully plastic, the constraint factor will generally be less than 3.0 (but not less than the elastic limit of 1.1).

Full plasticity on fused silica at 100nm



Overall Summary - continued

- For Hertzian contacts, plasticity is governed by geometry, properties, AND applied force. Indents become more severe with indentation depth.
- For Hertzian contacts, plasticity initiates below the surface when the mean pressure of the contact (i.e. the “hardness”) exceeds the yield stress for the material.
- For sharp contacts, plasticity is governed by geometry and properties, NOT applied force. Indents do not become more severe with indentation depth. Yield occurs from the outset, if it occurs at all.

Overall Summary

- Awareness of thresholds for plasticity help us to cause or avoid plasticity as desired.
- In instrumented indentation, “Hardness” is always defined as the mean pressure of the contact, whether or not the contact causes plasticity.
- Hardness has a quantitative relationship to yield stress ONLY if the contact causes plasticity. Otherwise, “hardness” depends only on elastic properties and geometry.
- For fully plastic contacts, $H \approx 3\sigma_y$ (only applies if the material has a well defined yield stress; i.e. metals.)
- For fully developed contacts $H = C\sigma_y$, where $1 < C < 3$.

Session 6: Basic instrumented indentation to measure hardness and Young's modulus
Wednesday, March 13, 2013, 11:00 (New York)

Abstract

In its most basic form, instrumented indentation involves pressing an indenter of known geometry into a test surface while continuously monitoring force and displacement. In this session, we review the basic test and analysis commonly known as the "Oliver-Pharr" method for measuring hardness and Young's modulus. The continuous measurement of force and displacement affords two important advantages over traditional hardness testing. First, the contact area can be analytically inferred and does not have to be optically measured. Second, the displacements measured during unloading manifest elastic recovery, and thus are a means for deriving Young's modulus by means of previously developed elastic contact models.

To register:

https://agilenteseminar.webex.com/agilenteseminar/onstage/g.php?p=117&t=m

Suggested reading for Session 6

Oliver, W.C. and Pharr, G.M., "An Improved Technique for Determining Hardness and Elastic-Modulus Using Load and Displacement Sensing Indentation Experiments," *Journal of Materials Research* **7**(6), 1564-1583, 1992.

Upcoming live workshops

- **May 1-3**: NMC 2013 10th International Workshop on Nanomechanical Sensing, Stanford, California (http://www.nmc2013.org)
- **June 2**: Theory & Practice of Instrumented Indentation Testing Course, Sunday, 1:00-5:00p.m., The Westin Lombard Yorktown Center, Lombard, Illinois. A short course offered in conjunction with the SEM 2013 Annual Conference & Exposition on Experimental and Applied Mechanics (http://sem.org/CONF-AC-TOP.asp)
- **June 25-26**: Nano Measure 2013—A symposium for sensing and understanding nano-scale phenomena, University of Warsaw, Poland, Abstract submission deadline: March 18, 2013 (www.nano-measure.com).

Thank you!